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A Method for Obtaining Darboux Transformations

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Abstract

In this paper we give a method to obtain Darboux transformations (DTs) of integrable equations. As an example we give a DT of the dispersive water wave equation. Using the Miura map, we also obtain the DT of the Jaulent-Miodek equation.

1 Introduction

For integrable equations which can be solved by the Inverse Scattering Transform, there exist Bäcklund transformations (BTs) [1]. These transformations were first discovered for the Sine-Gordon equation at the end of the 19th century. Usually they are treated as nonlinear superpositions, which allow one to create new solutions of a nonlinear evolution equation from a finite number of known solutions. In practice, BTs are not very straightforward to apply in the construction of multisolutions. On the other hand, the Darboux transformation (DT) is a very convenient way of constructing new solutions of nonlinear integrable equations [2]; the algorithm is purely algebraic and can be continued successively. Therefore, it is interesting to transform BTs into DTs.

Many integrable equations of the form

$$u_t = K(u) \tag{1.1}$$

possess the recursion operator Φ with the property called hereditary symmetry [3, 4, 5, 6], and they possess a Lax pair

$$\begin{aligned} &\Phi\sigma = \lambda\sigma, \\ &\sigma_t = K_u\sigma. \end{aligned} \tag{1.2}$$

Here K_u is the Fréchet derivative of K with respect to u. Two interesting questions are raised: "How is the DT related to the Lax pair (1.2)?" and "What happens to the symmetry σ under a BT?"

In this paper, we will study the above problems. Section 2 gives the general method to obtain DTs of integrable equations by using symmetry. As an example, Section 3 gives the DT of the dispersive water equation. In Section 4, we obtain the DT of the Jaulent-Miodek equation by using the Miura map. These DTs are not easily obtained by other well-known methods.

2 The Method

Suppose that the equation (1.1) has a BT of the form

$$B(u, u[1]) = 0. \tag{2.1}$$

Now, we suppose that

$$u_u[1][\sigma_1] = 0, \tag{2.2}$$

which means that the symmetry σ_1 is transformed into 0 under the BT, where σ_1 is the eigenfunction of (1.2) with $\lambda = \lambda_1$.

Then taking Fréchet derivative of $B = 0$, we have

$$B_u[\sigma_1] = B_u[\sigma_1] + B_{u[1]}[u_u[1]\sigma_1] = 0. \tag{2.3}$$

Fuchssteiner and Aiyer have showed that the KdV equation, the Burgers equation, and the CDGSK equation admit this relation [7, 8, 9].

This formula gives the transformation relation between u, σ_1 and $u[1]$:

$$u[1] = F(u, \sigma_1). \tag{2.4}$$

At this point we can directly check whether (2.4) is a BT. If so, we can conclude that (2.3) is true, and we also have the transformation for eigenfunctions

$$\sigma[1] = u_u[1][\sigma] = F_u[\sigma] + F_{\sigma_1}[\sigma_{1u}\sigma]. \tag{2.5}$$

Relations (2.4) and (2.5) is called the DT of (1.1).

Remark 2.1. Here we give a method to calculate $\sigma_{1u}[\cdot]$. Because $\Phi\sigma_1 = \lambda\sigma_1$, we have

$$\sigma_{1u}[\cdot] = -(\Phi - \lambda_1)^{-1}\Phi_u[\cdot]\sigma_1.$$

We can apply the factorization method to calculate $(\Phi - \lambda_1)^{-1}$.

Remark 2.2. When (2.4) and (2.5) is the DT

$$\Phi(u[1])\sigma[1] = \lambda\sigma[1],$$

and (2.5) is the symmetry of $u_t[1] = K(u[1])$, the result [10] shows that (2.4) and (2.5) is a DT for the hierarchy $u_t = \Phi^n K(u)$.

Remark 2.3. Relation (2.4) reveals the connection among the BT, symmetry, and strong symmetry operator. We conjecture that (2.4) may be right for all equations which possess

strong symmetry operators. For some $(1+1)$-dimension equations ($u_t = K(u)$), their usual Lax pair

$$L\phi = \lambda\phi, \qquad \phi_t = A\phi$$

can be transformated into

$$\Psi\sigma = \mu\sigma, \qquad \sigma_t = K_u\sigma$$

by a transformation $\sigma = f(\phi)$. We can then obtain the DT with respect to the usual Lax pair.

3 The DT of the Dispersive Water Wave Equation

In this section we study the dispersive water wave equation (DWW) [11, 12]

$$v_t = K(v), \tag{3.1}$$

where

$$v = (q, r)^T,$$
$$K(v) = \left(\frac{1}{2}(2qr - q_x)_x, \frac{1}{2}(r_x + r^2 + 2q)_x\right)^T$$

and T denotes the transpose of vectors. The DWW equation was studied systematically by Kupershmidt [11].

System (3.1) has the following Lax representation [11].

$$L\phi = \lambda\phi, \tag{3.2}$$

$$\phi_t = \frac{1}{2}(L^2)_+\phi, \tag{3.3}$$

in which

$$L = D + (D - r)^{-1} \circ q, \qquad D = \frac{\partial}{\partial x}, \qquad D^{-1} \circ D = D \circ D^{-1} = 1$$

and $(\cdot)_+$ is the projection to the purely differential part of the operator, $(L^2)_+ = D^2 + 2q$. Here we denote the operator A acting on the operator B by $A \circ B$, and the operator A action on a function f by Af.

The system (3.1) possesses a strong symmetry operator

$$\Phi(v) = \begin{pmatrix} -D + r & 2q + q_x \circ D^{-1} \\ 2 & D + D \circ r \circ D^{-1} \end{pmatrix}. \tag{3.4}$$

So (3.1) is the following integrable condition

$$\Phi\sigma = \lambda\sigma, \tag{3.5}$$

$$\sigma_t = K_v\sigma, \tag{3.6}$$

where $\sigma = (\sigma_1, \sigma_2)^T$.

It is difficult to get a DT of (3.1) with respect to (3.2), (3.3). In fact, we did not find any DT for (3.2), (3.3) until now.

Let us turn to (3.5), (3.6).

Theorem 3.1. *Let $\sigma_{,1} = (\sigma_{1,1}, \sigma_{2,1})^T$ denote the solution of (3.5), (3.6) with $\lambda = \lambda_1$. We then have the DT*

$$q[1] = q - \left(\frac{D^{-1}\sigma_{1,1}}{D^{-1}\sigma_{2,1}}\right)_x, \tag{3.7}$$

$$r[1] = r + \left(\ln \frac{D^{-1}\sigma_{1,1} + \sigma_{2,1}}{D^{-1}\sigma_{2,1}}\right)_x, \tag{3.8}$$

$$\sigma_1[1] = \sigma_1 - \left(\frac{B}{D^{-1}\sigma_{2,1}}\right)_x, \tag{3.9}$$

$$\sigma_2[1] = \sigma_2 - \left(\frac{B + \sigma_2 D^{-1}\sigma_{2,1}}{D^{-1}\sigma_{1,1} + \sigma_{2.1}}\right)_x, \tag{3.10}$$

where

$$B = D^{-1}(\sigma_1 D^{-1}\sigma_{2,1} + \sigma_2 D^{-1}\sigma_{1,1}),$$

that is, (3.7) and (3.8) is a new solution of (3.1). Moreover, (3.7), (3.8) and (3.9), (3.10) satisfy (3.5), (3.6). Furthermore, this is the DT of the hierarchy $v_t = \Phi(v)^n K(v)$.

Proof. (i) From practice, we first suppose $q[1] = q + (ln\phi_1)_{xx}$ is a part of the BT. Substituting $\phi_1 = e^{D^{-2}(q[1]-q)}$ into (3.2) with $\lambda = \lambda_1$, we find

$$(D^{-1}(q[1] - q))^2 + q[1] + \lambda_1 r - (r + \lambda_1)D^{-1}(q[1] - q) = 0. \tag{3.11}$$

From (2.3) we find

$$-(2D^{-1}\sigma_{1,1} + \sigma_{2,1})D^{-1}(q[1] - q) + (r + \lambda_1)D^{-1}\sigma_{1,1} + \lambda_1\sigma_{2,1} = 0.$$

On the other hand, (3.5) gives

$$2D^{-1}\sigma_{1,1} + \sigma_{2,1} = (\lambda_1 - r)D^{-1}\sigma_{2,1}.$$

These two identities imply (3.7).

Suppose $(q[1], r[1])$ satisfy (3.1), then

$$2(q[1] - q)_t = (2q[1]r[1] - 2qr - (q[1] - q)_x)_x, \tag{3.12}$$

$$-2\left(\frac{D^{-1}\sigma_{1,1}}{D^{-1}\sigma_{2,1}}\right)_t = 2q[1]r[1] - 2qr + \left(\frac{D^{-1}\sigma_{1,1}}{D^{-1}\sigma_{2,1}}\right)_{xx}. \tag{3.13}$$

Using (3.6), we have

$$\begin{aligned} 2q[1]r[1] - 2qr + \left(\frac{D^{-1}\sigma_{1,1}}{D^{-1}\sigma_{2,1}}\right)_{xx} &= -\frac{1}{D^{-1}\sigma_{2,1}}(2\sigma_{1,1}r + 2q\sigma_{2,1} - \sigma_{1,1x}) \\ &\quad + \frac{D^{-1}\sigma_{1,1}}{(D^{-1}\sigma_{2,1})^2}(\sigma_{2,1x} + 2r\sigma_{2,1} + 2\sigma_{1,1}). \end{aligned} \tag{3.14}$$

We note that (3.5) yields

$$r=\lambda_1-\frac{\sigma_{2,1}+2D^{-1}\sigma_{1,1}}{D^{-1}\sigma_{2,1}}, \tag{3.15}$$

$$q=\frac{1}{(D^{-1}\sigma_{2,1})^2}(\sigma_{1,1}D^{-1}\sigma_{2,1}+(D^{-1}\sigma_{1,1})^2). \tag{3.16}$$

Substituting the above two identities and (3.7) into (3.14), we obtain, after some calculations, (3.8).

We can easily prove that (3.7) and (3.8) satisfy (3.5) and (3.6), so (3.7), (3.8) is a BT of (3.1).

(ii) From (3.15) and (3.16)

$$v_{\sigma_1}D=\frac{1}{D^{-1}\sigma_{2,1}}\begin{pmatrix} D+\dfrac{2D^{-1}\sigma_{1,1}}{D^{-1}\sigma_{2,1}} & -\dfrac{\sigma_{1,1}}{D^{-1}\sigma_{1,1}}-2\left(\dfrac{D^{-1}\sigma_{1,1}}{D^{-1}\sigma_{2,1}}\right)^2 \\ -2 & -D+\dfrac{\sigma_{2,1}+2D^{-1}\sigma_{1,1}}{D^{-1}\sigma_{2,1}} \end{pmatrix}.$$

Now, we solve the equation

$$v_\sigma D\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}=\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$$

that is,

$$a_{1x}+2\frac{D^{-1}\sigma_{1,1}}{D^{-1}\sigma_{2,1}}a_1-\left(\frac{\sigma_{1,1}}{D^{-1}\sigma_{2,1}}+2\left(\frac{D^{-1}\sigma_{1,1}}{D^{-1}\sigma_{2,1}}\right)^2\right)a_2=\sigma_1D^{-1}\sigma_{2,1}, \tag{3.17}$$

$$-2a_1-a_{2x}+\left(\frac{\sigma_{2,1}+2D^{-1}\sigma_{1,1}}{D^{-1}\sigma_{2,1}}\right)a_2=\sigma_2D^{-1}\sigma_{2,1}. \tag{3.18}$$

$(3.17)+(3.18)\dfrac{D^{-1}\sigma_{1,1}}{D^{-1}\sigma_{2,1}}$, we find

$$a_1-a_2\frac{D^{-1}\sigma_{1,1}}{D^{-1}\sigma_{2,1}}=B$$

with

$$B=D^{-1}((D^{-1}\sigma_{2,1})\sigma_1+\sigma_2D^{-1}\sigma_{1,1}).$$

Hence

$$a_1=B-(D^{-1}\sigma_{1,1})\left(2D^{-1}\frac{B}{D^{-1}\sigma_{2,1}}+D^{-1}\sigma_2\right),$$

$$a_2=-2D^{-1}\sigma_{2,1}D^{-1}\frac{B}{D^{-1}\sigma_{2,1}}-D^{-1}\sigma_2D^{-1}\sigma_{2,1}.$$

Now we can calculate $\sigma[1]$ from (3.7), (3.8):

$$\begin{aligned}
\sigma[1] &= v_v[1]\sigma \\
&= \sigma + D\begin{pmatrix} \dfrac{-1}{D^{-1}\sigma_{2,1}} & \dfrac{D^{-1}\sigma_{1,1}}{(D^{-1}\sigma_{2,1})^2} \\ \dfrac{1}{D^{-1}\sigma_{1,1}+\sigma_{2,1}} & \dfrac{D}{D^{-1}\sigma_{1,1}+\sigma_{2,1}} - \dfrac{1}{D^{-1}\sigma_{2,1}} \end{pmatrix} D^{-1}(v_{\sigma_1})^{-1}\sigma \\
&= \sigma + D\begin{pmatrix} -\dfrac{1}{D^{-1}\sigma_{2,1}} & \dfrac{D^{-1}\sigma_{1,1}}{(D^{-1}\sigma_{2,1})^2} \\ \dfrac{1}{D^{-1}\sigma_{1,1}+\sigma_{2,1}} & \dfrac{D}{D^{-1}\sigma_{1,1}+\sigma_{2,1}} - \dfrac{1}{D^{-1}\sigma_{2,1}} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\
&= \sigma - D\begin{pmatrix} \dfrac{B}{D^{-1}\sigma_{2,1}} \\ \dfrac{B+\sigma_2 D^{-1}\sigma_{2,1}}{\sigma_{2,1}+D^{-1}\sigma_{1,1}} \end{pmatrix}.
\end{aligned}$$

This completes the proof.

Remark 3.1. Let $w = \sigma_x$, then (3.5), (3.6) can be written in a more simple form:

$$\begin{aligned}
&-w_{xx} + rw_{1x} + 2qw_{2x} + q_x w_2 = \lambda w_{1x}, \\
&2w_{1x} + (w_{2x} + rw_2)_x = \lambda w_{2x}, \\
&2w_{1t} = 2rw_{1x} + 2w_{2x}q - w_{1xx}, \\
&2w_{2t} = 2w_{2xx} + 2rw_{2x} + 2w_{1x}.
\end{aligned}$$

The DT given in Theorem 3.1 becomes

$$\begin{aligned}
q[1] &= q - \left(\frac{w_{1,1}}{w_{2,1}}\right)_x, \\
r[1] &= r + \left(\ln\frac{w_{1,1}+w_{2,1x}}{w_{2,1}}\right)_x, \\
w[1] &= w - \begin{pmatrix} \dfrac{B}{w_{2,1}} \\ \dfrac{B + w_{2,1x}(1)w_{2,1}}{w_{1,1}+w_{2,1x}} \end{pmatrix},
\end{aligned}$$

with

$$B = D^{-1}(w_{2,1}w_{1x} + w_{2x}w_{1,1}).$$

4 The DT of the Jaulent-Miodek equation

The Jaulent-Miodek equation takes the form [13, 14]

$$u_t = H(u) = \Psi(u)u_x, \tag{4.1}$$

where

$$u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \qquad \Psi(u) = \begin{pmatrix} 0 & \left(\frac{1}{4}D^3 + \frac{1}{2}u_0 \circ D + \frac{1}{2}D \circ u_0\right) \circ D^{-1} \\ 1 & \left(\frac{1}{2}u_1 \circ D + \frac{1}{2}D \circ u_1\right) \circ D^{-1} \end{pmatrix}$$

and Ψ is a strong symmetry operator. So (4.1) possesses the Lax pair

$$\Psi(u)\psi = \lambda\psi, \tag{4.2}$$

$$\psi_t = H_u\psi. \tag{4.3}$$

Usually (4.1) is derived from the following spectral problem [12, 13, 14]:

$$L\phi = \phi_{xx} + (u_0 + \lambda u_1)\phi = \lambda^2\phi, \tag{4.4}$$

where the time evolution of the wave function ϕ has the form

$$\phi_t = P\phi = \left(\frac{1}{2}p \circ D - \frac{1}{4}p_x\right)\phi, \tag{4.5}$$

with $p = 2 + \lambda u_1$. Then

$$L_t - [P, L] = p_x L$$

gives rise to (4.1). The BT of (4.1) was given by Tu [13]. It is not easy to apply this BT to construct new solutions.

An invertible Miura map [12]

$$q = u_0 + \frac{1}{4}(u_1)^2 - \frac{1}{2}u_{1x}, \tag{4.6}$$

$$r = u_1 \tag{4.7}$$

brings (4.1) into the DWW (3.1).

The Miura map (4.6), (4.7) gives the relation of the eigenfunctions between (4.2), (4.3) and (3.5), (3.6):

$$\psi = u_v\sigma = \begin{pmatrix} 1 & \frac{D}{2} - \frac{r}{2} \\ 0 & 1 \end{pmatrix}\sigma \tag{4.8}$$

$$= \begin{pmatrix} \sigma_1 + \frac{\sigma_{2x}}{2} - \frac{r}{2}\sigma_2 \\ \sigma_2 \end{pmatrix}. \tag{4.9}$$

Therefore,

$$D^{-1}\sigma_1 = D^{-1}\psi_1 - \frac{\psi_2}{2} + \frac{D^{-1}(u_1\psi_2)}{2} = \left(\lambda - \frac{u_1}{2}\right)D^{-1}\psi_2 - \frac{\psi_2}{2},$$

$$u_1[1] = u_1 + \left(\ln\frac{D^{-1}\sigma_{1,1} + \sigma_{2,1}}{D^{-1}\sigma_{2,1}}\right)_x = u_1 + E, \tag{4.10}$$

where

$$E = \left(\ln\left(\lambda_1 - \frac{u_1}{2} - \frac{\psi_{2,1}}{2D^{-1}\psi_{2,1}}\right)\right)_x.$$

$$\begin{aligned} u_0[1] - u_0 &= (q[1] - q) + \frac{1}{2}(r[1] - r)_x - \frac{1}{4}(r[1] - r)(r[1] + r) \\ &= \left(\frac{1}{2}u_1 + \frac{\psi_{2,1}}{2D^{-1}\psi_{2,1}}\right)_x + \frac{1}{2}E_x - \frac{1}{4}E(E + 2u_1), \end{aligned} \tag{4.11}$$

$$\begin{aligned} B &= D^{-1}(\sigma_1 D^{-1}\sigma_{2,1} + \sigma_2 D^{-1}\sigma_{1,1}) \\ &= (D^{-1}\psi_{2,1})\left(\left(\lambda_1 - \frac{u_1}{2}\right) D^{-1}\psi_2 - \frac{\psi_2}{2}\right) \\ &- D^{-1}\left(\psi_{2,1}\left(\lambda_1 - \frac{u_1}{2}\right) D^{-1}\psi_2 - \psi_2\left(\lambda_1 - \frac{u_1}{2}\right) D^{-1}\psi_{2,1}\right), \end{aligned}$$

$$\psi_2[1] = \psi_2 - F, \tag{4.12}$$

$$\psi_1[1] = \psi_1 - \left(\frac{B}{D^{-1}\psi_{2,1}}\right)_x - \frac{F_x}{2} - \frac{1}{2}(E\psi_{2,1} - F(u_1 + E)), \tag{4.13}$$

with

$$F = D\left(\frac{B + \psi_2 D^{-1}\psi_{2,1}}{\left(\lambda_1 - \frac{1}{2}u_1\right) D^{-1}\psi_{2,1} + \frac{1}{2}\psi_{2,1}}\right).$$

Theorem 4.1. *Suppose $(u, \psi_{,1})$ satisfies (4.2), (4.3) with $\lambda = \lambda_1$, then the transformation defined by (4.10), (4.11), (4.12), (4.13) is a DT of (4.2), (4.3).*

5 Conclusion

In this paper, we have presented a method to obtain DTs of integrable equations. This method can be apply to the DWW equation, the KdV equation, a shallow water equation [15] and other integrable equations. We think the relation (2.3) is very important because it reveals the relation between BT, symmetry, and strong symmetry of the corresponding equation. We hope that there will be further study in this direction.

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